EJERCICIO A

PROBLEMA 3. Encontrar razonadamente el punto de la curva $y = \frac{1}{1+x^2}$ en el que la recta tangente a la curva tiene pendiente máxima y calcular el valor de esta pendiente. *(3,3 puntos)*

Solución:

La pendiente de la recta tangente a la curva $y = \frac{1}{1+x^2}$ *en cualquier punto de ella se obtiene calculando y´*

$$y' = \frac{-2x}{\left(1+x^2\right)^2}$$

Sea $m = \frac{-2x}{\left(1+x^2\right)^2}$ *, queremos que sea máxima, debe ser m´ = 0 y m´´ < 0.*

Calculemos m´ y m´´

$$m' = \frac{-2\left(1+x^2\right)^2 - (-2\,x)\,2\,\left(1+x^2\right)2\,x}{\left(1+x^2\right)^4} = \frac{-2\left(1+x^2\right)+8x^2}{\left(1+x^2\right)^3} = \frac{6x^2-2}{\left(1+x^2\right)^3}$$

$$m'' = \frac{12x\left(1+x^2\right)^3 - (6x^2-2)\,3\left(1+x^2\right)^2 2x}{\left(1+x^2\right)^6} = \frac{12x\left(1+x^2\right)-(6x^2-2)\,6x}{\left(1+x^2\right)^4} = \frac{12x+12x^3-36x^3+12x}{\left(1+x^2\right)^4} =$$

$$= \frac{-24x^3+24x}{\left(1+x^2\right)^4} = \frac{24x(1-x^2)}{\left(1+x^2\right)^4}$$

$$m' = 0 \;\rightarrow\; \frac{6x^2-2}{\left(1+x^2\right)^3} = 0 \;\rightarrow\; 6x^2-2=0 \;\rightarrow\; 6x^2 = 2 \;\rightarrow\; x^2 = \frac{2}{6} \;\rightarrow$$

$$\rightarrow\; x^2 = \frac{1}{3} \;\rightarrow\; x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}$$

$$Para \quad x = \frac{1}{\sqrt{3}} \;\rightarrow\; m'' = \frac{24\frac{1}{\sqrt{3}}\left(1-\left(\frac{1}{\sqrt{3}}\right)^2\right)}{\left(1+\left(\frac{1}{\sqrt{3}}\right)^2\right)^4} = \frac{24\frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)}{\left(1+\frac{1}{3}\right)^4} = \frac{\frac{24}{\sqrt{3}}\,\frac{2}{3}}{\left(\frac{4}{3}\right)^4} > 0 \;\Rightarrow\; mínimo$$

$$Para \quad x = \frac{-1}{\sqrt{3}} \;\rightarrow\; m'' = \frac{24\frac{-1}{\sqrt{3}}\left(1-\left(\frac{-1}{\sqrt{3}}\right)^2\right)}{\left(1+\left(\frac{-1}{\sqrt{3}}\right)^2\right)^4} = \frac{24\frac{-1}{\sqrt{3}}\left(1-\frac{1}{3}\right)}{\left(1+\frac{1}{3}\right)^4} = \frac{\frac{-24}{\sqrt{3}}\,\frac{2}{3}}{\left(\frac{4}{3}\right)^4} < 0 \;\Rightarrow\; máximo$$

$$Para \quad x = \frac{-1}{\sqrt{3}} \;\rightarrow\; y = \frac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} = \frac{1}{1+\frac{1}{3}} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$$m = \frac{-2\,\frac{-1}{\sqrt{3}}}{\left(1+\left(\frac{-1}{\sqrt{3}}\right)^2\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(1+\frac{1}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\left(\frac{4}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{16}{9}} = \frac{18}{16\sqrt{3}} = \frac{9}{8\sqrt{3}} = \frac{9\sqrt{3}}{8\,.\,3} = \frac{3\sqrt{3}}{8}$$

En conclusión, el punto de curva $y = \dfrac{1}{1+x^2}$ *en que la recta tangente a ella tiene pendiente máxima es* $\left(\dfrac{-1}{\sqrt{3}}, \dfrac{3}{4}\right)$ *y esta pendiente es* $\dfrac{3\sqrt{3}}{8}$.